From coupled networks of systems to networks of states in phase space

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Abstract

Dynamical systems on graphs can show a wide range of behaviours beyond simple synchronization - even simple globally coupled structures can exhibit attractors with intermittent and slow switching between patterns of synchrony. Such attractors, called heteroclinic networks, can be well described as networks in phase space and in this paper we review some results and examples of how these robust attractors can be characterised from the synchrony properties as well how coupled systems can be designed to exhibit given but arbitrary network attractors in phase space.

1 Introduction to dynamics of coupled networks of systems

Many applications, especially in biology and theoretical physics, can be usefully described in terms of coupled dynamic cells, where the coupling between the cells describes a network of interconnections. This is especially the case for models of neural systems where individual cells (neurons) behave in relatively simple ways, but the system as a whole can have very complex dynamical, and indeed computational, behaviour: see for example the reviews [23, 8]. There is a large literature on the emergent attracting dynamics of coupled systems: this examines in particular the appearance of different types of synchrony, bifurcations, oscillatory behaviour, thermodynamic properties of large systems and noise-induced phenomena; see for example [62].

However, there is a relatively small literature that recognises that attracting dynamics may robustly include attracting (nonchaotic) networks in phase space that are quite curious in their structure and properties. These properties are particularly useful in describing intermittent behaviour of networks and in ascribing some computational functions of networks to these attractors: see for example

[64, 59, 75, 73, 16, 57, 66]. Going beyond cyclic networks in phase space one needs to understand how a trajectory can approach the same "node" in the network but then switch direction to follow different outgoing connections: see for example [19, 5, 53, 13]. Our aim in this paper is to review some of the theory behind this material, both for symmetric and non-symmetric coupled systems, and to provide some extensions of examples in the literature.

The first, and arguably most famous, example of a robust heteroclinic network consists of three equilibria and heteroclinic orbits connecting them (see Fig. 1): this was first found and investigated by May and Leonard [55], in a Lotka-Volterra model describing population dynamics, and later by Busse and Clever in the context of Rayleigh-Benard convection [18]. Structural stability for such a system on an invariant sphere was proved by dos Reis [24, Example (4.1d), p641] in the context of equivariant vector fields on two-manifolds. Guckenheimer and Holmes brought the example to the attention of a wider community by showing the appearance of the cycle through a symmetry breaking bifurcation at a symmetric equilibrium [36].

These attracting heteroclinic networks are curious in a number of ways - although they may be attractors in the sense of asymptotic stability, unlike chaotic or periodic attractors they typically have no single trajectory that is dense within the attractor. Hence they do not support a natural ergodic measure [32]. Worse than that, for typical observables the time averages do not converge, so useful measures for chaotic dynamics (e.g. Lyapunov exponents) cannot be used without extreme care, as they simply do not converge [40, 10]. Such attractors can be robust (structurally stable) in the context of coupled systems and so their existence cannot be ignored. Since structural stability may not be familiar to a wide audience interested in coupled dynamical systems, we review some relevant concepts within the paper.

Our paper is organized as follows: we start in section 2 by a discussion of properties of heteroclinic network attractors in phase space, in particular heteroclinic networks and some contexts in which they may be robust. These contexts include symmetric networks and nonsymmetric networks with specific coupling types, as discussed in section 3, and coupled oscillators [37, 47, 59, 43], as reviewed in [10]. An obvious question is how the structure of a network attractor in phase space relates to the structure of the network of coupled cells - so far progress has mostly been by examples than by general constructions. As an alternative approach, section 4 discusses how heteroclinic (and excitable) networks with arbitrary structure in phase space can be designed for appropriate coupled systems. Finally section 5 highlights some open problems relating to network attractors in phase space.

2 Networks of states in phase space

Just as network dynamical systems consist of nodes (i.e. the systems that are coupled together) and edges (representing the influence of one system on another) one can think of invariant structures and in particular attractors in phase space as having network structure if there are nodes (i.e. simple invariant sets) and edges (representing trajectories that connect one node to another). To avoid confusion, in the former context we will refer to cells and edges, and in the latter context to nodes and connections.

In this work we restrict our discussion primarily to heteroclinic connections between equilibria, and the extension to periodic orbits is relatively direct, but note

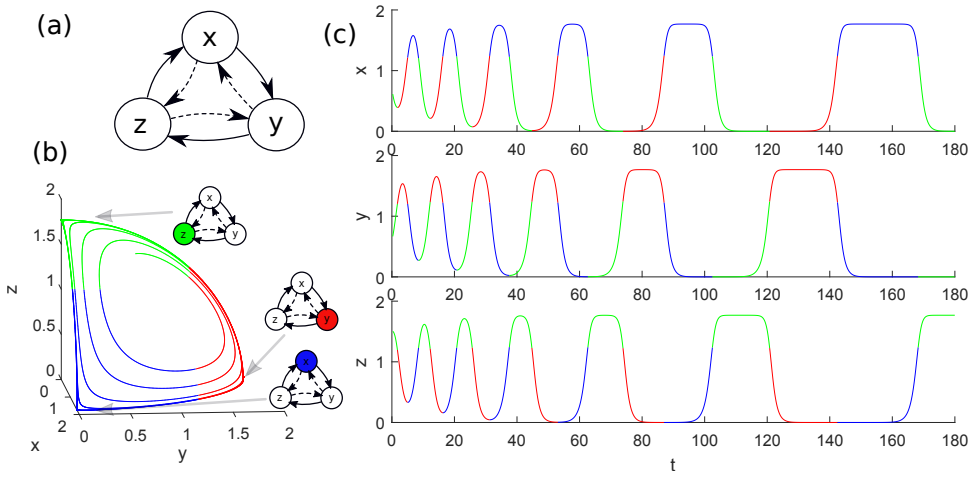


Figure 1: (a) Coupled cell structure of the Guckenheimer-Holmes system, see Example 2. (b) The Guckenheimer-Holmes heteroclinic cycle in phase space as the limiting set of a trajectory between the saddle equilibria $(x, y, z) = (\xi, 0, 0)$, $(0, \xi, 0)$ and $(0, 0, \xi)$ for some $\xi > 0$, with active cells along the cycle indicated. Note that this is part of a larger network of twelve connections between the six equilibria $(\pm\xi, 0, 0)$, $(0, \pm\xi, 0)$ and $(0, 0, \pm\xi)$; typical initial conditions limit to one of eight possible cycles. (c) Corresponding time-series showing asymptotic slowing down as the trajectory approaches the heteroclinic cycle.

that other types of networks have been considered, for example networks between chaotic invariant sets [21, 26, 42, 73] or unstable attractors for semiflows [14, 45].

2.1 Heteroclinic networks and cycles

Assume throughout that M is a smooth finite dimensional connected manifold, and $(\varphi_t)_{t\in\mathbb{R}}$ a smooth flow of M, typically corresponding to the solution $x(t) = \varphi_t(x_0)$ of an ordinary differential equation

$$\frac{d}{dt}x = f(x)$$

for $x \in M$. Although many of the concepts discussed in this review apply to discrete and hybrid dynamical systems or semiflows as well, we focus our attention on smooth flows.

A heteroclinic connection, formally defined below in Def. 2.1, is a solution trajectory $\gamma : \mathbb{R} \to M$ of a dynamical system that limits onto one flow-invariant set A backwards in time, and another a flow-invariant set B forwards in time. Recall that the ω-limit set and α-limit set of a point $x \in M$ are, respectively,

$$\omega(x) = \bigcap_{t>0} \overline{\{\varphi_s(x) : s > t\}}, \quad \alpha(x) = \bigcap_{t<0} \overline{\{\varphi_s(x) : s < t\}}.$$

where $\overline{A}$ denotes the closure of A.

The stable set $W^s(A)$ of a flow-invariant set $A \subset M$ is the set of all points with ω-limit set contained in A:

$$W^s(A) = \{x \in M : \omega(x) \subset A\},$$

and the unstable set $W^u(A)$ of A is the set of all points with α-limit set contained in A:

$$W^u(A) = \{x \in M : \alpha(x) \subset A\}.$$

If A is a hyperbolic invariant set, $W^s(x)$ and $W^u(x)$ are manifolds for each $x \in A$, called the stable and unstable manifolds of A at x, cf. [60] and [70, Chapter 6]. If A is a normally hyperbolic invariant manifold, e.g. a hyperbolic equilibrium or periodic orbit, then $W^s(A)$ and $W^u(A)$ are manifolds, that furthermore persist under perturbations [38, 74].

With these concepts at hand, the definition of a heteroclinic connection can now be phrased more precisely in terms of stable and unstable sets. A flow-invariant set is called recurrent if it is the ω-limit of some trajectory within the set. Let γ be as above, and let A and B be compact flow-invariant recurrent subsets of M.

Definition 2.1. The solution trajectory γ is a *heteroclinic connection* from A to B, written $A \xrightarrow{\gamma} B$, if A and B are disjoint and $\gamma(\mathbb{R}) \subset W^u(A) \cap W^s(B)$. If $A = B$, and $\gamma(\mathbb{R}) \nsubseteq A$, then γ is said to be a *homoclinic connection*.

A heteroclinic network is a graph in phase space where the nodes are invariant sets and the edges are heteroclinic connections. Such networks provide a mathematical framework for dynamical transitions between different states of a system. There are various definitions in the literature that vary somewhat depending on purpose.

By *node*, we will mean a compact flow-invariant recurrent subset, as in the definition above. The following definition is similar to [30, Definition 1.1], but allows for more general nodes.

Definition 2.2. A *heteroclinic network* is a bounded set $\Sigma = \mathcal{A} \cup \Gamma \subset M$ consisting of a finite set of pairwise disjoint nodes $\mathcal{A}$ and a set Γ of heteroclinic connections, with $\gamma \in W^u(A) \cap W^s(B)$ for nodes $A \neq B$ in $\mathcal{A}$ for all $\gamma \in \Gamma$, such that for all ordered pairs of nodes $A, B \in \mathcal{A}$, there is a sequence $A = A_0 \xrightarrow{\gamma_1} A_1 \xrightarrow{\gamma_2} \ldots \xrightarrow{\gamma_k} A_k = B$ of connections and nodes in Σ from A to B.

Definition 2.3. A *heteroclinic cycle* is a heteroclinic network where there is an ordering of the nodes $\mathcal{A} = \{A_1, \ldots, A_n\}$ such that there exists a connection $A_i \xrightarrow{\gamma} B$ if and only if $B = A_{i+1}$ (subscripts considered modulo n).

Remark 1. This definition of a heteroclinic cycle allows arbitrarily many connections between A_i and A_{i+1}, and so differs from [30], but agrees with, for example, [9, Definition 2.10]. Note that a cycle can contain a proper subset of all heteroclinic connections from A_i to A_{i+1}, and need not contain all unstable sets of its nodes.

As the nodes are not limited to equilibria or periodic orbits, our definition includes a variety of networks between chaotic or other types of invariant sets: see for example [21, 26, 42, 14, 45]. See also [9] for an intrinsic definition of heteroclinic networks.

It follows from the definition that heteroclinic networks are flow-invariant. Furthermore, they are transitive as graphs (in particular connected), and so every

heteroclinic network is the union of heteroclinic cycles. Since we allow for a continuum of connections between nodes (as is possible for example if the network contains a hyperbolic equilibrium with an unstable manifold of dimension greater than one [6]) a heteroclinic network is not necessarily compact. However, if the number of connections is finite, the network is compact. To distinguish from the more complicated cases alluded to, a heteroclinic network is said to be *simple* if all nodes are equilibria with one-dimensional unstable manifolds. Note that simple heteroclinic networks are compact.

Remark 2. In equivariant dynamics, it is natural to regard symmetry related nodes and connections as essentially the same, and to consider networks and cycles modulo the group action. A heteroclinic cycle as defined above might therefore be called homoclinic if all nodes are symmetry related [48]. Similarly, a heteroclinic network may be a homoclinic or heteroclinic cycle in the symmetric sense, depending on which nodes and connections are the same modulo the group action. If nothing else is stated, we will use the terms in the usual, non-symmetric sense. See Example 2 for a brief discussion regarding the Guckenheimer-Holmes cycle.

2.2 Genericity, transversality and robustness

Let X be a topological space. A set $Y \subset X$ is *residual* if it contains a countable intersection of open and dense subsets of X, and X is a called a *Baire space* if all residual sets are dense. A property $\mathcal{P}$ is called *generic* if it holds for all elements in a residual set. In our setting, let $\mathcal{X}(M)$ denote the space of vector fields on M, equipped with the Whitney C^∞-topology, and note that $\mathcal{X}(M)$ is a Baire space in this topology [33]. We say a property $\mathcal{P}$ of vector fields is *robust* if for any $F \in \mathcal{X}$ with the property, there is an open neighbourhood $U \subset \mathcal{X}$ of F such that $\mathcal{P}$ holds on U. Of particular interest to us will be the case where $\mathcal{P}$ is the property of having some given heteroclinic network in phase space.

A concept closely related to robustness is structural stability: a vector field $F \in \mathcal{X}(M)$ is *structurally stable* if there exists a neighbourhood U of F in $\mathcal{X}(M)$ such that any vector field G in U is topologically equivalent to F, i.e. there exists a homeomorphism $h : M \to M$ sending the orbits of F to the orbits of G and preserving the time orientation [60, Chapter 1].

The simplest example of a heteroclinic connection is that between hyperbolic equilibria. In this case, the equilibria have global stable and unstable manifolds, as noted above. A heteroclinic connection from p to q corresponds then to a non-trivial intersection of the unstable manifold of p with the stable manifold of q. As in most cases of interest, heteroclinic connections are trajectories contained in the intersection of invariant manifolds, an understanding of such intersections is central in the study of heteroclinic dynamics. In particular, for equilibria and periodic orbits, the concept of transversality is of importance to understand if an intersection is robust or not to perturbations.

Two submanifolds N_1 and N_2 of a manifold M are said to intersect *transversely* if at any point p in their intersection, the tangent spaces of N_1 and N_2 span the tangent space of M, i.e. $T_pN_1 + T_pN_2 = T_pM$. Note that if the intersection is empty, N_1 and N_2 are trivially transverse. A key feature of transversality is that it is a generic property - any non-transverse intersection of submanifolds may be made transverse by an arbitrarily small smooth perturbation. This is the content

of Thom's Transversality Theorem [33]. Furthermore, the Kupka-Smale Theorem ([60, Chapter 3], [61]) states that generically in $\mathcal{X}(M)$, equilibria and periodic orbits are hyperbolic and their stable and unstable manifolds intersect transversely. Consequently, only heteroclinic connections contained in a transverse intersection of unstable and stable manifolds persist under perturbations in the class of general smooth vector fields.

Using this, it is easy to see that heteroclinic cycles of equilibria are not robust in $\mathcal{X}(M)$ [11]. Assume that Σ is a robust heteroclinic cycle in a phase space with dimension n and nodes $\{x_1, ..., x_k\}$ that are equilibria of the flow. A minimal requirement for robustness in this context is hyperbolicity of the nodes of Σ, and so the equilibria $x_1, ..., x_k$ must be hyperbolic. The connections lie in intersections of stable and unstable manifolds, and by the Kupka-Smale Theorem these are generically transverse. Hence, as generic properties are dense in $\mathcal{X}(M)$, the intersections $W^u(x_i) \cap W^s(x_{i+1})$ must all be transverse for Σ to be robust. This, together with the fact that all $W^u(x_i) \cap W^s(x_{i+1})$ by assumption contain a connection, and so $\dim(W^u(x_i) \cap W^s(x_{i+1})) \geq 1$, yields

$$\dim(W^u(x_i)) + \dim(W^s(x_{i+1})) \geq n + 1.$$

It follows that

$$\sum_{j=1}^{k} [\dim(W^u(x_i)) + \dim(W^s(x_{i+1}))] \geq k(n+1).$$

On the other hand, since $\dim(W^u(x_i)) + \dim(W^s(x_i)) = n$ for all hyperbolic equilibria x_i, we have

$$\sum_{j=1}^{k} [\dim(W^u(x_i)) + \dim(W^s(x_{i+1}))] = kn.$$

This is clearly a contradiction, and so we have shown that heteroclinic cycles of equilibria are not robust to general perturbations. Being unions of heteroclinic cycles, the same applies to heteroclinic networks.

While heteroclinic networks cannot appear robustly for general vector fields, systems with additional structure may support robust heteroclinic networks. Typically, this is due to the presence of invariant subspaces - an intersection of invariant manifolds need not be generically transverse within a restricted class of vector fields if it is contained in a subspace that is invariant for all vector fields in the class [28, 24, 48].

To illustrate, let F be a vector field, in a class $\mathcal{Y} \subset \mathcal{X}(M)$ with the property that $P \subset M$ is an invariant subspace for all vector fields in $\mathcal{Y}$. Let p and q be equilibria of F lying in P, and assume p is a saddle for the restriction of F to P, and q a sink. Now, saddle-sink connections are robust, so if there is a heteroclinic connection from p to q, it will be robust to perturbations in the class $\mathcal{Y}$. This depends ultimately on the persistence of invariant manifolds of hyperbolic equilibria: if F' in $\mathcal{Y}$ is sufficiently close to F, it will have hyperbolic equilibria p' and q' with invariant manifolds in P arbitrarily close to those of p and q.

Historically, the two types of systems supporting heteroclinic networks given the most attention are equivariant dynamical systems, and Lotka-Volterra systems

in mathematical biology and game theory [48, 40, 39]. In the former, invariant subspaces arise as fixed point spaces in phase space of the system symmetries [48, 26], and in the latter as extinction planes, i.e. subspaces with one or more variables equal to zero [40]. Other contexts in which heteroclinic phenomena appear robustly are Hamiltonian systems (e.g. [46]), reversible dynamical systems, and, as will be our focus, network dynamical systems.

2.3 Stability of heteroclinic networks

Many heteroclinic networks can be attractors, in various senses. However, the stability analysis is intricate, as they often display weaker forms of stability than asymptotic stability, and we will only very briefly discuss the topic here, and mainly give references. Note that these references concern heteroclinic networks of equilibria (for stability conditions for cycles of chaotic sets, see [26, Chapter 7]).

Asymptotic stability of robust heteroclinic cycles in equivariant systems has been systematically studied by Krupa and Melbourne in [49] and [50], where sufficient (and in some cases, necessary) conditions in terms of eigenvalues of equilibria are given. Intuitively, a cycle is asymptotically stable if the contracting eigenvalues dominate the expanding and transverse eigenvalues of equilibria around the cycle.

Cycles and networks that do not contain all unstable manifolds of their nodes, are not asymptotically stable, or even Lyapunov stable. Still, they may have other rather strong stability properties, such as essential asymptotic stability, defined by Melbourne [56], which corresponds to asymptotic stability up to a set of very small measure: see [17, 52, 63, 20] for more details and examples. For heteroclinic networks, one may study the stability of cycles within the network, or stability of the network as a whole. For such investigations, see Kirk and Silber [44], Brannath [17] and Castro and Lohse [20].

3 Networks in phase space for coupled cell systems

In the dynamics of coupled systems, there are two distinct but related mechanisms through which invariant subspaces typically arise - network symmetry and coupling structure, both giving rise to robust synchrony of groups of cells. As part of the study of equivariant dynamical systems, symmetric coupled systems have thus far received the most attention in the literature. Indeed, the existence of robust synchrony through coupling structure in non-symmetric coupled systems was not discovered until the seminal papers by Stewart, Golubitsky et al. [71, 34, 35].

In this section we include a review of some literature on coupled cell systems in general, and how this relates to the existence of heteroclinic networks in phase space for such systems.

3.1 Formalisms for coupled cell systems

Dynamical systems on graphs or networks may be formalized as so called coupled cell systems. These are systems of interacting dynamical units (cells), connected according to an underlying network structure, with directed edges indicating dynamical influence of one cell on another. As two cells might respond differently to the same input, and a cell might be influenced by two cells in different ways, it is

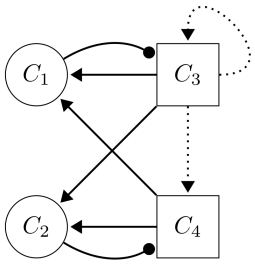


Figure 2: A coupled cell network with two cell types and three edge types, graphically indicated by different box and arrow styles. As in 1. of Def. 3.1, edges of the same type have equivalent sources and targets. Note also that the input sets $I(c_1)$ and $I(c_2)$, and $I(c_3)$ and $I(c_4)$ respectively, consist of the same number of edges of each type, as required by 2. of Def. 3.1.

natural to distinguish cells and edges of different types. Furthermore, it is useful to require that cells of the same type receive in some sense the same input.

Denote by $I(c)$ the *input set* of c, consisting of those edges targeting c, and recall the standard definition of source and target functions s and t for directed multigraphs, sending a directed edge to its source cell and target cell respectively.

Definition 3.1. (cf. [35, 34, 31, 58]). A *coupled cell network* $\mathcal{N}$ (hereafter simply network if the context is clear) is a directed multigraph $(\mathcal{C}, \mathcal{E}, s : \mathcal{E} \to \mathcal{C}, t : \mathcal{E} \to \mathcal{C})$ equipped with equivalence relations $\sim_{\mathcal{C}}$ on the set of cells $\mathcal{C}$, called *cell type*, and $\sim_{\mathcal{E}}$ on the set of edges $\mathcal{E}$, called *edge type*. We require the following, and refer to Fig. 2 for an illustration.

1. If $e \sim_{\mathcal{E}} f$, then $s(e) \sim_{\mathcal{C}} s(f)$ and $t(e) \sim_{\mathcal{C}} t(f)$.
2. If $c \sim_{\mathcal{C}} d$, then there exists a bijection $\beta : I(c) \to I(d)$ that preserves edge types, i.e. $e \sim_{\mathcal{E}} \beta(e)$ for all $e \in I(c)$, where $I(c) = \{e \in \mathcal{E} : t(e) = c\}$.

The maps β are called *input isomorphisms*. They define an algebraic structure, similar to a group, called a groupoid. Referred to as the network symmetry groupoid, it has been given much attention in the early literature on coupled cell networks [71, 35, 34].

Next, we define coupled cell systems - dynamical systems defined through vector fields, called admissible vector fields, compatible with a given coupled cell networks. We will restrict attention to smooth flows and vector fields, and all phase spaces will be smooth manifolds, typically finite-dimensional vector spaces.

Definition 3.2 (Coupled cell system)**.** (cf. [35, 34, 31, 58]). A *coupled cell system* $\mathcal{F}$ on a network $\mathcal{N}$ consists of:

1. A phase space M_c for each cell $c \in \mathcal{C}$, with $M_c = M_d$ whenever $c \sim_{\mathcal{C}} d$. By $x_c \in M_c$ we will denote the state variable associated to cell c. The *total phase space* of $\mathcal{F}$ is $M = \prod_{c \in \mathcal{C}} M_c$.
2. A cell component function

$$f_c : \prod_{e \in I(c)} M_{s(e)} \to TM_c$$

for each cell $c \in \mathcal{C}$, modelling the cells' dynamical response to inputs, where TM_c is the tangent bundle of M_c. These functions are required to satisfy

$$f_c \circ \beta^* = f_d$$

for all input isomorphisms $\beta : I(c) \to I(d)$, where

$$\beta^* : \prod_{e \in I(d)} M_{s(e)} \to \prod_{e \in I(c)} M_{s(e)}$$

is defined as $(\beta^* x)_{s(e)} = x_{s(\beta(e))}$.

3. An *admissible vector field* $F : M \to TM$, defined component-wise as $(Fx)_c = f_c\left(\prod_{e \in I(c)} x_{s(e)}\right)$ for cell component functions as above.

Remark 3. It will be assumed throughout that every cell effects its own state, in the sense that f_c depends on x_c. This internal coupling will be represented by always giving x_c as the first argument in f_c, and is suppressed in diagrams of network architecture.

Remark 4. The input isomorphism requirement means in particular that cell component functions are symmetric in inputs of the same edge type. In practice, it is often convenient to put an order on the edge types, and then always give the arguments of the component functions in this order. The requirement can then be reduced to the statement that $f_c = f_d$ whenever $c \sim_{\mathcal{C}} d$, with f_c symmetric in inputs corresponding to the same edge type.

Remark 5. Conventions differ between authors (and even between papers with the same author). In contrast to the original definition of Stewart and Golubitsky et al. (e.g. [35]), but in line with others ([3, 29, 58]), we require cells of the same type to be input isomorphic.

Example 1. The admissible vector fields for the network of Fig. 2 are of the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_3, x_4}) \quad \dot{x}_3 = g(x_3, x_1, x_3) \\ \dot{x}_2 &= f(x_2, \overline{x_3, x_4}) \quad \dot{x}_4 = g(x_4, x_2, x_3), \end{aligned}$$

where x_i is the state associated to cell c_i, f and g are generically different functions. Note the internal dynamical dependence of each cell on its own state, see Remark 3. The notation $\overline{x, y}$ denotes symmetry in the variables x and y, see Remark 4.

Many of the networks studied in the literature are *identical cell networks*, meaning that all cells are of the same type. Furthermore, the networks in our later examples will have *asymmetric inputs*: there is at most one edge of each edge type targeting a cell.

3.2 Heteroclinic networks in symmetric systems

A network symmetry, also called a global symmetry, is a permutation of the cells that commutes with all admissible vector fields. As such, the fixed point space of a network symmetry is flow-invariant. Note that there are other types of possible

symmetries for a coupled cell system, such as input symmetries induced by multiple inputs being of same edge type, or other local symmetries of the component functions such as in Example 2 below.

For networks with symmetries, methods of equivariant dynamical systems used to study robust heteroclinic networks can be applied. This research has mostly focused on heteroclinic cycles, but there are exceptions, such as the network defined and analysed by Kirk and Silber in [44]. Two phenomena in equivariant dynamics have been shown to create robust heteroclinic cycles; symmetry breaking bifurcations [36, 27, 41] and forced symmetry breaking [52, 51, 69]. For an early review of the research on robust heteroclinic cycles in symmetric systems, see [48], and also chapter 7 of [26] for a treatment of symmetrically coupled cell systems.

Furthermore, some of the qualitative results on heteroclinic dynamics developed for equivariant dynamical systems, such as the stability results discussed in section 2.3, remain valid in systems without symmetry. More precisely, those results in the equivariant theory of robust heteroclinic networks that depend only on the existence of an invariant subspace structure do not crucially require symmetry, but rather use symmetry as a natural setting in which such invariant subspaces arise.

Example 2. The Guckenheimer-Holmes system on $M = \mathbb{R}^3$ is the flow induced by

$$\begin{aligned} \dot{x} &= x - ax^3 - x(by^2 + cz^2) \\ \dot{y} &= y - ay^3 - y(bz^2 + cx^2) \\ \dot{z} &= z - az^3 - z(bx^2 + cy^2) \end{aligned} \tag{1}$$

where a, b, c are real parameters. The system is symmetric under the maps $x \mapsto -x$, $y \mapsto -y$ and $z \mapsto -z$ (and cyclic permutations of the coordinates) and therefore the coordinate planes are invariant under the flow of (1). For an open set of parameter values, the system has an asymptotically stable heteroclinic cycle between equilibria $(\pm\xi, 0, 0)$, $(0, \pm\xi, 0)$ and $(0, 0, \pm\xi)$ [36, 26, 48]. In Fig. 1 (b,c), the robust cycle is shown as the ω-limit set of the initial value $(0.7, 0.6, 1.5)$, and parameter values $a = 0.32, b = 0.1, c = 0.58$, giving $\xi = 1.7677$.

Strictly speaking, the cycle is only heteroclinic in the usual sense if we regard just the part shown in Fig. 1 (b), and is asymptotically stable only if we restrict the dynamics to this flow-invariant octant, but not in the whole phase space. However, the cycle generates a heteroclinic network with twelve connections and six equilibria (via the system symmetries) which is asymptotically stable in $\mathbb{R}^3$. Indeed, since all nodes and all connections are symmetry related, it is a homoclinic cycle in the symmetric sense.

The Guckenheimer-Holmes system can be interpreted as a system of three coupled cells with internal dynamics given by a pitchfork bifurcation normal form and cubic coupling. For a schematic picture, see Fig. 1 (a). With the same coupling, but replacing the internal dynamics with a Chua circuit or a Lorenz attractor, it is possible to get the same heteroclinic cycle but with chaotic sets instead of equilibria [21]. Note that the cycle is robust to perturbations respecting the symmetries of (1), but not to those only respecting the network structure.

3.3 Synchrony in non-symmetric coupled cell systems

There are various notions of synchrony and synchronization in nonlinear dynamics [62, 23]. Here, we will say two cells are synchronous if their states have identical time

evolutions. Such synchrony between cells may or may not depend on the specific choice of vector field. The existence of robust synchrony, that depends only on the network structure of a system and therefore persists under structure preserving perturbations, has a significant impact on network dynamics.

A fully or partially synchronous state of a coupled cell system is described by equalities of groups of cell states, and corresponds to a partition of the set of cells. Note that the states of two cells may be compared only if their phase spaces are the same. Any partition P of $\mathcal{C}$ corresponds to an equivalence relation $\sim_P$, and P is said to *refine* $\sim_\mathcal{C}$ if $c \sim_P d$ implies $c \sim_\mathcal{C} d$ [34]. Such a partition defines a subspace

$$\Delta(P) = \{x \in M : x_c = x_d \text{ if } c \sim_P d\}.$$

Definition 3.3. [34]. For a partition P as above, the subspace $\Delta(P)$ is called a *synchrony subspace* if it is flow-invariant for all admissible vector fields.

Remark 6. It is perhaps worth emphasizing that in the definition above, invariance under all admissible vector fields means a synchrony subspace is invariant under any vector field compatible with the network structure, including choice of phase space.

Network symmetries induce robust synchrony: the fixed point space of a network symmetry is clearly a synchrony subspace. However, a coupled cell network can have many more synchrony subspaces than those induced by symmetries. Indeed, there are many networks with no symmetries that possess a rich synchrony subspace structure.

A combinatorial and very useful characterization of synchrony subspaces is given in terms of so called balanced partitions: A partition P is *balanced* if whenever $c \sim_P d$, there exists an input isomorphism $\beta : I(c) \to I(d)$ such that $s(e) \sim_P s(\beta(e))$ for every edge $e \in I(c)$. In other words, every pair of synchronous cells must receive the same number of inputs of each edge type, from all groups of synchronous cells. As first proved in [71], $\Delta(P)$ is a synchrony subspace if and only if P balanced.

Since synchrony subspaces are flow-invariant, restricting the dynamics of a coupled cell system to a synchrony subspace gives a new dynamical system, which again has the structure of a coupled cell system, called the quotient network with respect to P.

Definition 3.4. For a balanced partition $P = \{P_1, ..., P_r\}$ of a network $\mathcal{N}$, the *quotient network* $\mathcal{N}_P$ is the network with cells $\{P_1, ..., P_r\}$ and arrows inherited from representatives of the equivalence classes P_i. More precisely, the arrows from P_i to P_j of type k are identified with the arrows of type k from the cells in P_i to a representative $c \in P_j$. Since P is balanced, this is independent of the choice of $c \in P_j$.

Due to the presence of synchrony subspaces, surprisingly complicated dynamics can occur even for small coupled cell networks. In terms of robust heteroclinic dynamics in non-symmetric systems, the main focus has been to explore existence of heteroclinic networks in coupled cell systems with few identical cells, without global and local symmetries, and with asymmetric inputs. For several examples, see [30].

As mentioned in section 3.2, symmetry breaking has been shown to create heteroclinic networks in symmetric systems. The general problem of synchrony breaking bifurcations in coupled cell systems has attracted attention in recent years,

see for example [4], [68] and [67], but there are few examples of synchrony breaking bifurcations to heteroclinic networks in non-symmetric systems in the literature. For one example, see [43], where a synchrony-breaking bifurcation in a system of four coupled oscillators is observed to create a heteroclinic ratchet - a heteroclinic network with all connections winding in one direction on the torus.

3.4 Heteroclinic cycles in a three cell network

Aguiar et al. show in [3] that, up to dynamical equivalence of networks (see [1, 2, 22] for definition and results), there are two identical cell networks with two asymmetric inputs supporting robust simple heteroclinic networks. We concentrate on the case $\mathcal{N}_3$ (notation from [3]) shown in Fig. 3. The admissible vector fields are of the form

$$\begin{aligned} \dot{x} &= f(x,y,z) \\ \dot{y} &= f(y,x,z) \\ \dot{z} &= f(z,y,x). \end{aligned} \tag{2}$$

We will write Δ for the fully synchronized diagonal $\{(x,...,x) \in M\}$, the maximal synchrony subspace. Note that in identical cell networks, Δ is always a synchrony subspace. As a demonstration of typical existence arguments for heteroclinic networks in coupled cell systems, we recount the arguments from [3] and [29] that $\mathcal{N}_3$ supports a robust heteroclinic cycle between two equilibria on Δ. We assume smooth vector fields, and that the cell dynamics of (2) are one-dimensional, i.e. that $M_c = \mathbb{R}$ for all $c \in C$; for topological reasons this is the most difficult case. Accordingly, let the cell phase spaces be $\mathbb{R}$, and the total phase space be $\mathbb{R}^3$.

By the characterization of synchrony subspaces through balanced partitions, one easily checks that $\mathcal{N}_3$ has non-maximal synchrony subspaces $\mathbf{P}_2 = \{(x,y,z) \in \mathbb{R}^3 : x = z\}$ and $\mathbf{P}_3 = \{(x,y,z) \in \mathbb{R}^3 : x = y\}$ (here adopting the notation from [29]). The argument splits into a local computation of eigenvalues, and a consideration of global obstructions due to network structure.

Let p be a point on the fully synchronized diagonal. Setting $\alpha = \frac{\partial f}{\partial x}(p), \beta = \frac{\partial f}{\partial y}(p), \gamma = \frac{\partial f}{\partial z}(p)$, the Jacobian of (2) at p is

$$J(p) = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \alpha & \gamma \\ \gamma & \beta & \alpha \end{pmatrix},$$

which has eigenvalues and eigenvectors

$$\begin{aligned} \mu_1 &= \alpha + \beta + \gamma, & v_1 &= (1,1,1), \\ \mu_2 &= \alpha - \beta, & v_2 &= (1, -\frac{\beta+\gamma}{\beta}, 1), \\ \mu_3 &= \alpha - \gamma, & v_3 &= (1, 1, -\frac{\beta+\gamma}{\gamma}), \end{aligned}$$

as is easily verified. Observe that $v_1 \in \Delta, v_2 \in \mathbf{P}_2$ and $v_3 \in \mathbf{P}_3$. By choosing α, β and γ, the eigenvalues μ_1, μ_2, μ_3 can be given any prescribed values (the assumption of asymmetric inputs is crucial: if, for example, f was symmetric in y and z, then $\beta = \gamma$). Hence f can be chosen such that $p, q \in \Delta$ are equilibria of (2), with one-dimensional unstable manifolds contained in $\mathbf{P}_2$ and $\mathbf{P}_3$ respectively, with p a sink

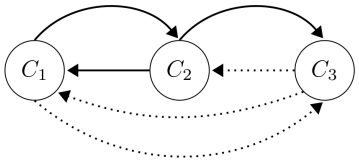


Figure 3: The three cell network architecture of $\mathcal{N}_3$ from [3]: note the presence of two edge types.

in $\mathbf{P}_3$ and q a sink in $\mathbf{P}_2$. These are the local properties needed in order to ensure that saddle-sink connections within the synchrony subspaces are possible.

Having checked that there are no local obstructions, we now turn our attention to global obstructions. Assume (2) is defined locally in a tubular neighbourhood D of Δ, with local properties as above. We wish to extend the vector field outside D. Consider the restrictions to $\mathbf{P}_2$ and $\mathbf{P}_3$.

$$\begin{array}{lll} \mathbf{P}_2: & \dot{x} = f(x,y,x) =: H_2(x,y), & \mathbf{P}_3: \quad \dot{x} = f(x,x,z) =: H_3(x,z) \\ & \dot{y} = f(y,x,x) =: V(x,y) & \qquad\quad \dot{z} = f(z,x,x) = V(x,z). \end{array} \tag{3}$$

As indicated, the "vertical components", i.e. the component functions for $\dot{y}$ and $\dot{z}$, are forced by the network structure to be identical. Let $\varphi : \mathbb{R} \to \mathbf{P}_2$ be a smooth curve from p to q that coincides with the unstable manifold of p in D. Then (re-parametrizing φ if necessary) (2) can be extend to have φ as a solution trajectory.

Similarly, let $\tilde{\varphi} : \mathbb{R} \to \mathbf{P}_3$ be a curve from q to p that coincides with the unstable manifold of q. Identifying points $(u,v,u) \in \mathbf{P}_2$ and $(u,u,v) \in \mathbf{P}_3$ with $(u,v) \in \mathbb{R}^2$, φ and $\tilde{\varphi}$ may or may not "intersect" in the sense that $\varphi(s) = \tilde{\varphi}(t)$ in $\mathbb{R}^2$ for some $s,t \in \mathbb{R}$. There are now two cases, depending on the eigendirections at p and q: either $\tilde{\varphi}$ may be taken to not intersect φ, in which case there are no global obstructions to extending (2) to have both desired connections, or $\tilde{\varphi}$ can be taken to intersect φ at only one point. At the intersection point, (3) implies $\tilde{\varphi}$ and φ need to have the same vertical component. This can be achieved by deforming and re-parametrizing $\tilde{\varphi}$ and φ (see [29] for details), using that the horizontal components H_2 and H_3 can be chosen independently. Thus in either case, there are no obstructions to extending the vector field as above, showing (2) supports a robust heteroclinic cycle between p and q with two connections.

Remark 7. By the same arguments as above, it is possible to have connections from p to q and from q to p on both sides of Δ, thus giving a heteroclinic cycle with four connections as in Example 4 and Fig. 4 below. The same method has also been used by Field in [29] to construct arbitrary heteroclinic networks in a family of coupled cell networks, of which $\mathcal{N}_3$ is the smallest member.

We mention briefly a method for finding more complex coupled cell networks with support for heteroclinic dynamics. Starting from a small network $\mathcal{N}$, one can identify those networks having $\mathcal{N}$ as a quotient. Such networks have been called *inflations* (for results and methods on enumerating and obtaining inflations of a given network, see [3, 4]). Say $\mathcal{N}$ supports a heteroclinic network Σ, and let $\mathcal{M}$ be an inflation of $\mathcal{N}$. The dynamics on $\mathcal{N}$ is the restriction to some synchrony subspace of $\mathcal{M}$, so Σ "lifts" to the dynamics of $\mathcal{M}$, i.e. $\mathcal{M}$ supports Σ. Note,

however, that there is no guarantee that, for example, simple networks lift to simple networks. It might be that eigenvalues transverse to the synchrony subspace into which Σ embeds cannot be chosen independently, forcing some unstable manifolds to have higher dimension. Typically, $\mathcal{M}$ will have additional synchrony subspaces, apart from those directly associated with $\mathcal{N}$, and hence may support more complex heteroclinic phenomena.

Example 3. A polynomial coupled cell system on $\mathcal{N}_3$ with a robust heteroclinic cycle between equilibria $p, q \in \Delta$ is given in [3], to which we refer for further details. The system is defined by (2) with component function

$$\begin{aligned} f(x,y,z) = x(1-x^2-y^2-z^2) + 0.13x(y-z) + \beta x^2(y+z) \\ + 0.18x^3(y-z) + \gamma(y^2-z^2) + \delta(y+z) + 0.25x^2(y^2-z^2), \end{aligned} \tag{4}$$

where $\gamma = 0.52, \delta = 0.17$ and $\beta \in (0, 0.4987)$.

At both p and q, only one branch of the unstable manifold is contained in the cycle. As noted in section 2.3, the cycle is thus not Lyapunov stable. We think of it as being "one-sided", in the sense that only one out of two "sides" of the unstable manifolds are involved in the cycle. Instead, these other branches limit unto other attracting equilibria. Nevertheless, numerical simulation suggests that the cycle is attracting for portions of phase space.

3.5 Construction of a two-sided cycle for $\mathcal{N}_3$

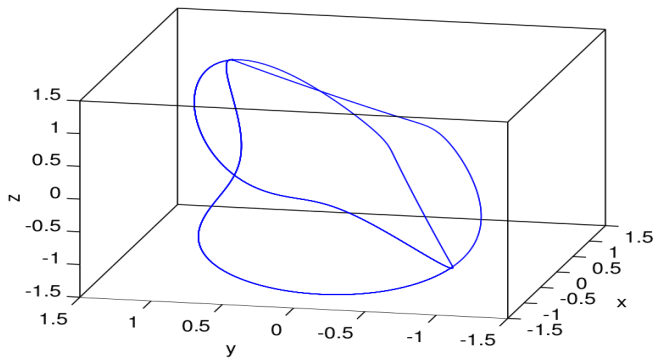
Example 3 gives a network with only one-sided connections: it was an open question in [3] as to whether this could be adapted to give two-sided connections. The next example is new, and shows that it is possible to find a polynomial vector field that answers this positively.

Example 4. Let

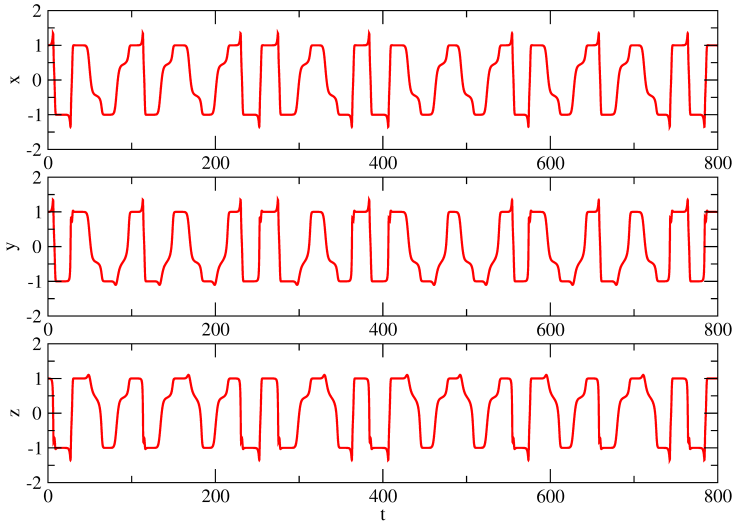
$$\begin{aligned} f(x,y,z) = \alpha x(1-x^2-y^2-z^2) + \beta x^2(y+z) + \gamma(y^2-z^2) \\ + \delta(y+z) - \epsilon(y+z)(x-y)(x-z) - \zeta(2x-y-z) \end{aligned} \tag{5}$$

where the parameters are chosen to be $\alpha = 1.05, \beta = 0.8, \gamma = 0.8, \delta = 0.25, \epsilon = 0.7, \zeta = 0.2$. With this choice of cell dynamics for the network $\mathcal{N}_3$, and for an open set of parameters, the coupled cell system (2) has two heteroclinic connections in each synchrony subspace, one on each side of Δ, between equilibria $p, q \in \Delta$. For the parameter choice above, $p = (-1,-1,-1)$ and $q = (1,1,1)$. The resulting heteroclinic cycle is shown in Fig. 4 (a). See Fig. 5 for nullclines and connections in the synchrony subspace $\mathbf{P}_2$, and Fig. 4 (b) for time-series of a trajectory along the cycle. Setting $\epsilon = \zeta = 0$ yields one-sided connections similar to those in Example 3.

The heteroclinic cycle is clearly compact. In contrast to the cycle in Example 3, it also contains all unstable manifolds of its nodes, and all intersections of stable and unstable manifolds. Hence the cycle is *clean*, in the terminology of Field [30], which is a necessary requirement for asymptotic stability. The contracting and expanding eigenvalues are $\lambda_c \approx -2.15$ and $\lambda_e \approx 1.05$, respectively, at both p and q, i.e. $|\lambda_e| < |\lambda_c|$. The condition in [49] shows the cycle is asymptotically stable.



(a)



(b)

Figure 4: (a) The heteroclinic cycle of Example 4 seen in phase space, with two nodes and four connections. (b) Time-series for a trajectory along the cycle in (a) with added independent noise of amplitude 10^{-7} in all components and initial condition $(x, y, z) = (1, 1.001, 0.999)$. Observe heteroclinic switching between two synchronized states $x = y = z = \pm 1$ but two types of switching owing to the two branches of the unstable manifold shown in Fig. 5: calculations using xppaut [25].

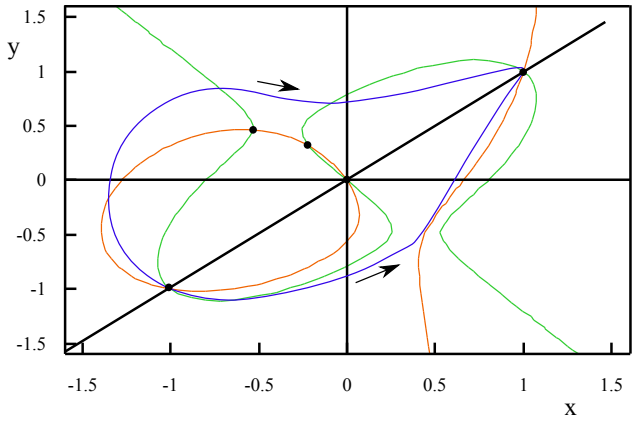


Figure 5: The system (2,5) in the synchrony subspace $\mathbf{P}_2$. Green lines show the nullcline for the y-component and red lines show the nullcline for the x-component, with parameters as in text. Blue curves are trajectories that approximate of the unstable manifold of $(x, y, z) = (-1, -1, -1)$: note that both branches of the unstable manifold are asymptotic to the sink at $(1, 1, 1)$. The numerical integration of (2,5) is performed using xppaut [25] and a Runge-Kutta integrator with time step 0.05.

4 Design of networks in phase space

The approaches outlined so far work from the premise that the coupled cell system has a given structure, and investigate the range of robust heteroclinic network structures that can arise from the global invariant structures (in particular the synchrony subspaces) in phase space.

An alternative approach is investigated in [7, 12, 29] where some arbitrary structure is assumed for the network in phase space, and candidates for cell networks that can realise this structure are presented. This may be a useful approach to understand the complexity of coupling structures in physical space that are needed to realise a given structure in phase space. Note that the constructions in [7, 12] use several types of cell.

Such constructions are of course highly non-unique: given a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with n_v vertices $\mathcal{V}$ and n_e edges $\mathcal{E}$, [7] give two constructions: one is a "simplex network" that gives a robust realization of $\mathcal{G}$ as a heteroclinic network on n_v cells, as long as $\mathcal{G}$ is one- and two-cycle free (i.e. $\mathcal{G}$ is such that all directed cyclic subgraphs are have three or more nodes). The same paper gives a robust realization of $\mathcal{G}$ as a heteroclinic network on $n_e + 1$ cells, as long as $\mathcal{G}$ is one-cycle free. More recently, [12] gives a construction on $n_v + n_e$ cells of two types, while [30] does this for $n_e + 1$ cells of the same type.

As highlighted in [12], there are a number of open questions about the dynamics of realization of arbitrary graphs as heteroclinic (or excitable) networks. Indeed, there seems to be a contrast between questions relating to the following categories:

- The noise-free dynamics, where the realization of a given graph is typically not asymptotically stable, if there is at least one node with a two-dimensional unstable manifolds. Moreover a connected network may contain a number of attractors that share common trajectories.

- The noisy dynamics, which seems to be much simpler, and clearly typical trajectories will explore the whole network even for arbitrarily small noise. However, the reasons for much of the simplicity may have quite subtle reasons [15]. Some specific conjectures about the behaviour of noisy network attractors in the limits of low levels of noise are discussed in [13].

4.1 A mixed heteroclinic/excitable network in phase space

The heteroclinic network of Kirk and Silber [44] consists of $n_v = 4$ nodes joined by two heteroclinic cycles of length 3, sharing a common edge and hence $n_e = 5$ edges. A minor generalization of the method of [12, Section 3] gives a network that is explicitly given in this case as

$$\begin{aligned}
\dot{p}_1 &= p_1(A_6(1-p^2) + A_4(p_1^2p^2 - p^4)) + A_5(y_5^2p_4^2 - y_1^2p_2p_1 - y_2^2p_3p_1)\\
\dot{p}_2 &= p_2(A_6(1-p^2) + A_4(p_2^2p^2 - p^4)) + A_5(y_1^2p_1^2 - y_3^2p_4p_2)\\
\dot{p}_3 &= p_3(A_6(1-p^2) + A_4(p_3^2p^2 - p^4)) + A_5(y_2^2p_1^2 - y_4^2p_4p_3)\\
\dot{p}_4 &= p_4(A_6(1-p^2) + A_4(p_4^2p^2 - p^4)) + A_5(y_3^2p_2^2 + y_4^2p_3^2 - y_5^2p_1p_4)\\
\dot{y}_1 &= G(y_1, A_1 - A_{21}p_1^2 + A_3(y^2 - y_1^2))\\
\dot{y}_2 &= G(y_2, A_1 - A_{22}p_1^2 + A_3(y^2 - y_2^2))\\
\dot{y}_3 &= G(y_3, A_1 - A_{23}p_2^2 + A_3(y^2 - y_3^2))\\
\dot{y}_4 &= G(y_4, A_1 - A_{24}p_3^2 + A_3(y^2 - y_4^2))\\
\dot{y}_5 &= G(y_5, A_1 - A_{25}p_4^2 + A_3(y^2 - y_5^2))
\end{aligned} \tag{6}$$

where p_i classify the nodes and y_i are active only during a transition between nodes. For (6) note that $G(y,\lambda) := -y((y^2-1)^2 + \lambda)$, $p^2 := p_1^2 + p_2^2 + p_3^2 + p_4^2$, $y^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2$, $p^4 := (p^2)^2$. We include independent standard Wiener additive noise processes of amplitudes η_p (resp. η_y) in each p (resp. y) directions. Some time-series shown in Fig. 6 are calculated using xppaut [25] for this system with parameters

$$\begin{aligned}
&A_1 = 0.5,\ A_{2k} = 1.5 - \nu_k,\ A_3 = 2,\\
&A_4 = 10,\ A_5 = 4,\ A_6 = 2,\ \eta_p = 0.001,\ \eta_y = 0.01.
\end{aligned} \tag{7}$$

Note that P_i, the ith state in this network, corresponds to the saddle equilibrium for the noise-free systsme where $p_i = 1$, $p_j = 0$ for $j \neq i$ and all $y_k = 0$: the itinerary consists of different p_i turning "on" while others remain "off", while the y_k become non-zero only during specific transitions. The small parameter ν_k governs the excitability of the network: $\nu_k < 0$ the kth connection is heteroclinic, while for $\nu_k > 0$ there is a threshold of size $O(\sqrt{\nu_k})$ that must be overcome to make the connection - the connection is excitable.

The full dynamical behaviour and attractor basin geometry for attractors of the Kirk-Silber network are subtle [44, 17, 19, 54]. This is particularly because this network has points with unstable manifolds of dimension greater than one, and there is an issue of "connection selection" [6] for typical trajectories. Rather than discussing these issues here, the inclusion of noise in (6) will regularise this behaviour in a way that allows one to understand the dynamics as approximated by a Markov chain on the network, with switching probabilities and waiting times at the nodes determined by the local response of the system to noise [13].

Figure 6 demonstrates that, in addition to the purely heteroclinic or purely excitable realisations of a given network discussed in [12, Section 4], the connections can be selectively set to be heteroclinic or excitable. More precisely, (a) shows typical time-series for (6) with parameters (7) and $\nu_k = -0.01$ for all k: all connections are heteroclinic and the influence of noise is to cause apparently random motion around the Kirk-Silber network. (b) of the same figure is the same except $\nu_2 = -0.04$ (making the probability of escape from the node P_1 to node P_2 is less likely) and $\nu_4 = 0.01$ (making the connection from node P_3 to node P_4 excitable rather than heteroclinic: note that in this case there are fewer escapes into node P_3, but when it does occur, in this case the mean time of escape is longer and the distribution of times has an exponential tail [13].

5 Discussion and open questions

In this paper we have reviewed some of the recent literature that discusses how networks in the coupling of systems (which represents the topology of coupling in physical space) can lead to the emergence of robust networks of connections within phase space, in particular for heteroclinic connections. Because these networks in phase space may be attractors as well as robust, they are in some sense just as real as periodic, chaotic or quasiperiodic attractors. Although we do not discuss applications in this paper, see for example [8, 64, 65, 72, 14, 16]. Note that these applications are typically for idealizations of neural systems and application to detailed spiking neuron models is the exception. It remains to be shown that these ideas can be convincingly applied in such detailed models.

We close with a brief summary of some open problems in this area.

- How can one efficiently determine the set of heteroclinic networks that can arise robustly in a given coupled system? [3] How does this depend on network topology, on phase space dimension and any other structures imposed (e.g. symmetries)?

- Can verifiable necessary and sufficient conditions for stability of such cycles be extended to networks? [49, 50] Note that this will depend on the definition of stability: for essential asymptotic stability this should be weaker than for asymptotic stability.

- For networks that are not just cycles, what must be in the closure of the simplest realization of an arbitrary graph? Under what, if any, circumstances do these forced nodes appear with non-zero asymptotic proportion of time in the limit of low noise? [7, 12, 13] What are the typical ω-limit sets in such networks?

- What are the best design strategies (in terms of minimal structures and/or minimal dimensions) to create robust networks in phase space? In particular, for a given coupled cell setting, what is the minimal number of coupled cells and/or connections between them that can be used to embed a given heteroclinic network in a robust manner? Probably the embeddings in [12, 30] are far from optimal in this sense.

- How do the above answers change if we consider nodes in phase space that are not just equilibria? [9]

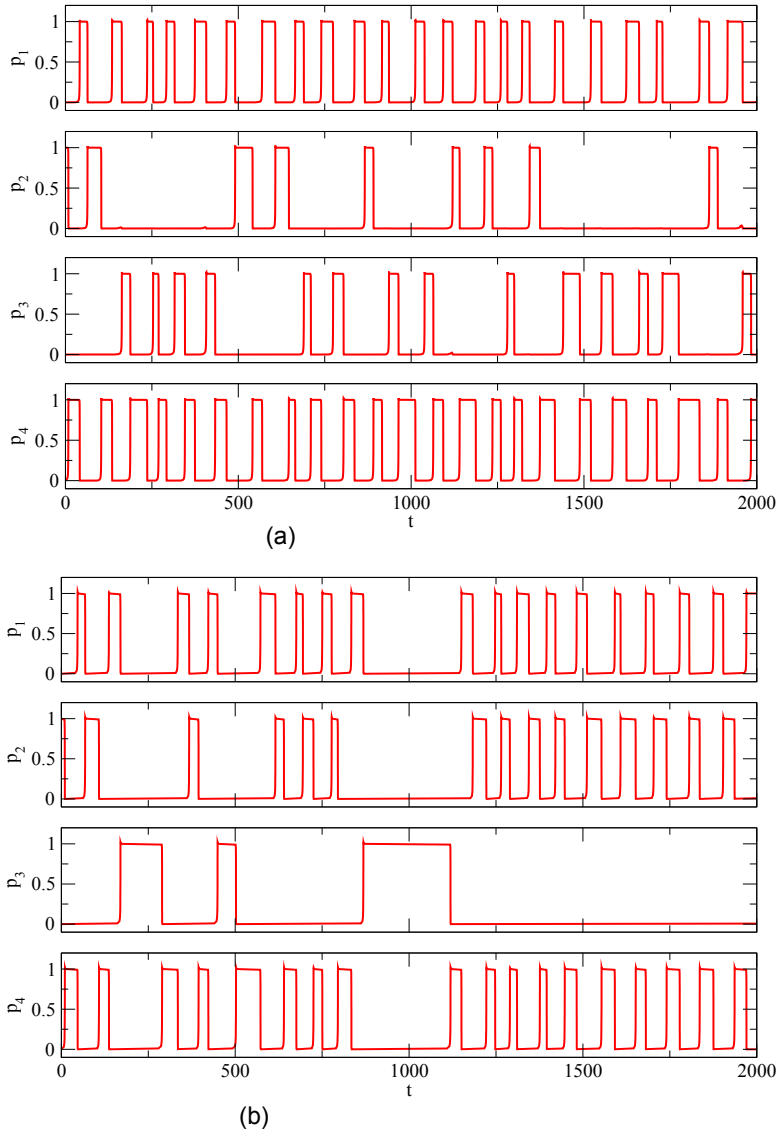


Figure 6: Time-series showing the p_i components of trajectories exploring a realization of the Kirk-Silber network using (6,7), with low amplitude noise. (a) shows a typical time-series for a heteroclinic network realization - note that the system moves around between four equilibria P_i, where $p_i = 1$ and $p_j = 0$ for $j \neq i$. The network is the union of the two cycles $P_1 \to P_2 \to P_4 \to P_1$ and $P_1 \to P_3 \to P_4 \to P_1$. (b) shows the case where one connection from P_1 has been made more unstable and in addition the connection from P_3 has been made excitable rather than heteroclinic. In particular, the noise-induced escape times for the heteroclinic transitions are fairly uniform while those for excitable transition are more widely distributed, consistent with exponential distribution.

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